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Internal gravity waves in a time-varying stratification

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The influence of slow time variations of the Brunt–Väisälä frequency N upon the energy of internal gravity waves is investigated. It is found that, when time variations in N are produced by a mean deformation field (reversible mean state), the wave energy can vary in either direct or inverse proportion, depending on the wavenumber orientation. When N changes owing to a certain type of irreversible process, the wave energy varies with only inverse proportionality.

The nocturnal planetary boundary layer (NPBL) provides an example where $N = N(z,t)$. The full initial/boundary-value problem for an $N(z,t)$ similar to the climatological mean for the NPBL is solved.

1. Introduction

The fundamental parameter governing the propagation of internal gravity waves is the Brunt–Väisälä frequency

$$N = \left(\frac{g}{\theta}\frac{\partial\theta}{\partial z}\right)^{\frac{1}{2}}, \tag{1.1}$$

where θ is the potential temperature. This quantity is predominantly a function of the altitude z in most geophysical applications. However, situations exist where N has a large temporal variation, and the consequences of this for internal gravity waves may be important. For example, Orlanski (1973) found that the diurnal stratification cycle of the planetary boundary layer could parametrically excite internal gravity waves of from small to mesoscale. McEwan & Robinson (1975) studied the wave–wave interaction process by considering the response of small-scale internal gravity waves to the periodic changes in stratification associated with a larger-scale wave motion.

Attention was confined in these studies to the long-time or cumulative effect of the periodic N^2 variation which produces instability. However, the local effect of a time-varying stratification can also add to (or subtract from) the wave energy without evoking the instability mechanism. The physical mechanism involved is in some respects similar to the effect produced on the energy of a string oscillation when the tension is altered.

A general theory for wave trains in slowly varying media was given by Bretherton & Garrett (1968); it was proved that the quantity termed the wave action is conserved. This result was demonstrated to hold in particular for internal gravity waves by Garrett (1968). Since the wave action is the local energy density E divided by the local Doppler-shifted frequency and this frequency is proportional to N, one might suspect

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that the behaviour of E as a functional of N is easily obtained. However, temporal variations of N are associated with a mean deformation field (in a conservative system) which also alters E. In § 2, Garrett's analysis is reconsidered and the wave energy equation is cast into a form where $E[N(t)]$ may be obtained given only the orientation of the wavenumber vector.

When $N(t)$ is produced by a certain type of irreversible process rather than a mean deformation field, the result concerning $E[N(t)]$ is very different. It is proved in § 2 that the quantity ωE is conserved in this case. One particular application of this result is to the nocturnal planetary boundary layer (NPBL), where the stratification builds up from zero after sunset. Some climatological averages are discussed briefly in § 3 and the solution to the full initial/boundary-value problem for internal gravity waves for an $N^2(z, t)$ similar to that observed is presented in § 4.

2. The wave energy equation

The derivation presented is the same as Garrett's except that here a diabatic heating (cooling) term is included in the mean state. The inclusion of this effect allows N to vary in response to not only mean deformation but also irreversible heat transfer.

The inviscid Boussinesq equations of motion are (see, for example, Batchelor 1953)

$$D\mathbf{u}/Dt = \sigma g \hat{\mathbf{e}}_z - \nabla\phi, \tag{2.1}$$

$$\nabla . \mathbf{u} = 0 \tag{2.2}$$

and

$$D\sigma/Dt = Q, \tag{2.3}$$

where $\mathbf{u} = (u, v, w)$ is the velocity vector, $\sigma = (\theta - \theta_a)/\theta_a$ is the deviation of the potential temperature from that of an adiabatic reference atmosphere, $\phi = (P - P_a)/\rho_a$ is the modified pressure and Q is the diabatic heating. The usual definitions

$$D/Dt = \partial/\partial t + \mathbf{u}.\nabla \quad \text{and} \quad \nabla = \partial/\partial x \hat{\mathbf{e}}_x + \partial/\partial y \hat{\mathbf{e}}_y + \partial/\partial z \hat{\mathbf{e}}_z$$

apply. The linearization of (2.1)–(2.3) is achieved by first partitioning u, σ and ϕ into a mean field and a perturbation therefrom. That is,

$$(u, v, w, \sigma, \phi) = (U, V, W, \Sigma, \Phi) + (u', v', w', \sigma', \phi'), \tag{2.4}$$

where all variables depend on x, y, z and t except for W† and Σ, which are allowed to vary only with z and t. In this analysis attention is restricted to the situation where the diabatic heating is uniform over a surface of constant height. This is possible in flows where the major heat-transport mechanism is small-scale turbulent convection. Given a stable stratification, the intensity of the turbulence is governed by the ambient wind shear, which, in turn, is determined by some influences which must be considered 'external' to (2.1)–(2.3). Our hypothesis is that, while displaced fluid parcels retain their potential temperature, they acquire the local turbulent intensity at the height to which they move. Therefore the heating rate Q is an *a priori* specified function of z and t which is unaffected by wave motion. In a more general context, a perturbation heating term should be included and the result concerning $E[N(t)]$ will depend on the mechanism of heat transfer. A detailed analysis of these effects will be found in a forthcoming article by Rotunno & Bretherton. Equation (2.4) is substituted into (2.1)–(2.3) and a set of equations involving only the mean quantities and another set involving both mean

† Garrett (1968) argued that $\partial W/\partial x$ and $\partial W/\partial y$ enter only at higher order.

and perturbation quantities emerge. The mean quantities are governed by (2.1)–(2.3) with $\mathbf{u}$, σ and ϕ replaced by $\mathbf{U}$, Σ and Φ. The perturbation equations are (dropping primes)

$$\frac{D_0 u}{Dt} + \mathbf{u}.\nabla U = -\frac{\partial \phi}{\partial x}, \tag{2.5}$$

$$\frac{D_0 V}{Dt} + \mathbf{u}.\nabla V = -\frac{\partial \phi}{\partial y}, \tag{2.6}$$

$$\frac{D_0 w}{Dt} + w\frac{\partial W}{\partial z} = -\frac{\partial \phi}{\partial z} + \sigma g, \tag{2.7}$$

$$\frac{D_0 \sigma}{Dt} + \frac{w}{g}N^2 = 0 \tag{2.8}$$

and

$$\nabla.\mathbf{u} = 0, \tag{2.9}$$

where $D_0/Dt = \partial/\partial t + U\,\partial/\partial x + V\,\partial/\partial y + W\,\partial/\partial t$.

The asymptotic solutions to (2.5)–(2.9) are obtained via the WKB approximation, in which all derivatives of $\mathbf{U}$ and N are neglected. The plane-wave solutions

$$\{\mathbf{u}, \sigma, \phi\} = \mathrm{Re}\,\{u_0, \sigma_0, \phi_0\} \exp i\beta(\mathbf{x}, t) \tag{2.10}$$

will satisfy (2.5)–(2.9) provided that the dispersion relation

$$\omega - \mathbf{U}.\mathbf{k} = \hat{\omega} = \pm\frac{N(k^2+l^2)^{\frac{1}{2}}}{(k^2+l^2+m^2)^{\frac{1}{2}}} \tag{2.11}$$

is satisfied. The quantities k, l, m and a are β_x, β_y, β_z and $-\beta_t$, respectively. The space–time dependence of these quantities is the subject of kinematic wave theory (see, for example, Whitham 1974, p. 380). The relationships among the amplitudes are

$$\left.\begin{aligned} u_0 &= \frac{k}{\hat{\omega}}\phi_0, \quad v_0 = \frac{l}{\hat{\omega}}\phi_0, \\ w_0 &= -\frac{(k^2+l^2)N^2}{m\hat{\omega}}\phi_0 \\ \text{and} \qquad \sigma_0 &= \frac{i(k^2+l^2)N^2}{m\hat{\omega}^2}\phi_0. \end{aligned}\right\} \tag{2.12}$$

The local energy density is defined as

$$E = \tfrac{1}{2}\overline{\mathbf{u}^2} + \tfrac{1}{2}g^2\overline{\sigma^2}/N^2 = \tfrac{1}{4}|\mathbf{u}_0|^2 + \tfrac{1}{4}g^2|\sigma_0|^2/N^2, \tag{2.13}$$

where the bar represents a local period average. The second-order correlations will be needed when the wave energy equation is considered and are given in terms of E as

$$\left.\begin{aligned} \overline{\phi\mathbf{u}} &= \hat{\mathbf{c}}E, \\ \overline{u^2} = \frac{k\hat{c}_1}{\hat{\omega}}E, \quad \overline{v^2} &= \frac{l\hat{c}_2}{\hat{\omega}}E, \quad \overline{w^2} = \frac{m\hat{c}_3}{\hat{\omega}}E + E, \\ \overline{uv} = \frac{k\hat{c}_3}{\hat{\omega}}E &= \frac{Em\hat{c}_1}{\hat{\omega}} - \frac{Ekm}{k^2+l^2}, \\ \overline{vw} = \frac{l\hat{c}_3}{\hat{\omega}}E &= \frac{Em\hat{c}_2}{\hat{\omega}} - \frac{Elm}{k^2+l^2}, \end{aligned}\right\} \tag{2.14}$$

where $\hat{\mathbf{c}}$ is the Doppler-shifted group velocity $\partial\hat{\omega}/\partial\mathbf{k}$, with components $(\hat{c}_1, \hat{c}_2, \hat{c}_3)$.

The energy equation is obtained by multiplying (2.5), (2.6) and (2.7) by u, v and w, respectively, then adding the resulting equations to obtain

$$\frac{D_0}{Dt}\frac{\mathbf{u}^2}{2}+u_i u_j\frac{\partial U_i}{\partial x_j}+\nabla.(\phi\mathbf{u}) = w\sigma g, \tag{2.15}$$

where (2.9) has been used. The term on the right-hand side of (2.15) may be evaluated by multiplying (2.8) by $\sigma g^2/N^2$:

$$w\sigma g = -\frac{1}{N^2}\frac{D_0}{Dt}\frac{g^2\sigma^2}{2} = -\frac{D_0}{Dt}\left(\frac{g^2\sigma^2}{2N^2}\right)-\frac{g^2\sigma^2}{N^2}\frac{1}{N}\frac{DN}{Dt}. \tag{2.16}$$

Hence the perturbation energy equation is

$$\frac{D_0}{Dt}\left(\frac{\mathbf{u}^2}{2}+\frac{g^2\sigma^2}{2N^2}\right)+u_i u_j\frac{\partial U_i}{\partial x_j}+\nabla.(\phi\mathbf{u})+\frac{g^2\sigma^2}{N^2}\frac{1}{N}\frac{D_0 N}{Dt} = 0. \tag{2.17}$$

The local period average of (2.17) is taken, and using the definition (2.13) and relations (2.14), the result is

$$\frac{dE}{dt}+E\nabla.\mathbf{c}+\frac{Ek_i\hat{c}_j}{\hat{\omega}}\frac{\partial U_i}{\partial x_j}+E\frac{\partial W}{\partial z}+\frac{E}{N}\frac{D_0 N}{Dt} = 0, \tag{2.18}$$

where $d/dt = \partial/\partial t+\mathbf{c}.\nabla$, where $\mathbf{c}$ is the group velocity. The first term is the rate of change of wave energy along a ray path, the second term represents the effect of converging (diverging) ray paths, the third and fourth terms are the effects of the working of the Reynolds stresses on the mean flow to extract (or add to) the energy of the basic state $\mathbf{U}$ and the last term is the effect of variable stability. Note that the last two terms and the deformation component of the third term cannot be independently specified if $Q = 0$.

Consider for the moment the situation $Q = 0$. The zeroth-order buoyancy equation is

$$D\Sigma/Dt = 0 \tag{2.19}$$

and it follows that

$$\frac{1}{N}\frac{D_0 N}{Dt} = -\frac{1}{2}\frac{\partial W}{\partial z}. \tag{2.20}$$

Combination of (2.18) and (2.20) verifies† that the conservation law

$$\frac{\partial}{\partial t}\left(\frac{E}{\hat{\omega}}\right)+\nabla.\left(\mathbf{c}\frac{E}{\hat{\omega}}\right) = 0 \tag{2.21}$$

applies for internal gravity waves and this was pointed out by Garrett. Remember that the goal of this section is to obtain E as a functional of N for both reversible and irreversible mean states. The Reynolds-stress term in (2.18) may be split into a part which depends on the mean shear and a part which depends on the mean deformation: it is

$$\frac{k\hat{c}_3}{\hat{\omega}}\frac{\partial U}{\partial z}+\frac{k\hat{c}_1}{\hat{\omega}}\frac{\partial U}{\partial x}+\frac{m\hat{c}_3}{\hat{\omega}}\frac{\partial W}{\partial z}.$$

† After the kinematical result

$$\frac{1}{\hat{\omega}}\frac{d\hat{\omega}}{dt} = -\frac{k_j\hat{c}_i}{\hat{\omega}}\frac{\partial U_j}{\partial x_i}+\frac{1}{N}\frac{D_0 N}{Dt}$$

has been applied. See Garrett's equation (1.10).

The last two terms may be combined since $k\hat{c}_1 = -m\hat{c}_3$ and $\partial U/\partial x = -\partial W/\partial z$. Using (2.20) the energy equation (2.18) becomes

$$\frac{dE}{dt} + E\nabla.\mathbf{c} + \frac{Ek\hat{c}_3}{\hat{\omega}}\frac{\partial U}{\partial z} + \left(\frac{4m^2}{m^2+k^2} - 1\right)\frac{E}{N}\frac{D_0 N}{Dt} = 0. \tag{2.22}$$

If α is defined as the angle between the wavenumber vector $\mathbf{k}$ and the horizontal, then the last term of (2.22) is

$$(4\sin^2\alpha - 1)\frac{E}{N}\frac{D_0 N}{Dt}. \tag{2.23}$$

The behaviour of E as a functional of N is as follows. If $\sin\alpha < \frac{1}{2}$, then increases (decreases) in N produce increases (decreases) in E. This implies that higher-frequency motion may be amplified, i.e. since $\hat{\omega}/N = \cos\alpha$ then $\sin\alpha < \frac{1}{2}$ implies that $\hat{\omega}/N > \frac{1}{2}\sqrt{3}$. If $\sin\alpha > \frac{1}{2}$, then the reverse is true.

When $Q \neq 0$, the results concerning the behaviour of E as a functional of N are very different. Consider a mean flow with zero deformation, i.e. $W = 0$, $U = U(z)$, $V = V(z)$. Then (2.18) becomes

$$\frac{dE}{dt} + E\nabla.\mathbf{c} + \frac{E\hat{c}_3}{\hat{\omega}}\left(\frac{k\partial U}{\partial z} + \frac{l\partial v}{\partial z}\right) + \frac{E}{N}\frac{D_0 N}{Dt} = 0, \tag{2.24a}$$

where N now changes solely in response to Q. Now when N increases (decreases), E decreases (increases). An important consequence of (2.24a) is that the conservation law (2.21) cannot be obtained. This result indicates the importance of knowing not only that N changes but also why it changes.

A conservation law analogous to (2.21) may be obtained. If $\mathbf{U} = 0$, then (2.24a) becomes

$$\frac{dE}{dt} + E\nabla.\mathbf{c} + \frac{E}{N}\frac{\partial N}{\partial t} = 0. \tag{2.24b}$$

A well-known result from the kinematic wave theory (see, for example, Whitham 1974, p. 383) is that

$$d\omega/dt = \partial\Omega/\partial t, \tag{2.25}$$

where in this case $\Omega = Nk/(k^2+m^2)^{\frac{1}{2}}$. Then

$$\frac{d\omega}{dt} = \frac{k}{(k^2+m^2)^{\frac{1}{2}}}\frac{\partial N}{\partial t} \tag{2.26}$$

and dividing both sides by ω gives

$$\frac{1}{\omega}\frac{d\omega}{dt} = \frac{1}{N}\frac{\partial N}{\partial t}. \tag{2.27}$$

Combination of (2.27) and (2.24b) yields the result

$$\partial(\omega E)/\partial t + \nabla(\mathbf{c}\omega E) = 0. \tag{2.28}$$

There is no claim of generality for (2.28), however it does seem to be the correct conservation law under the circumstances. In the next section an example of geophysical relevance and one where exact solutions are obtained is presented.

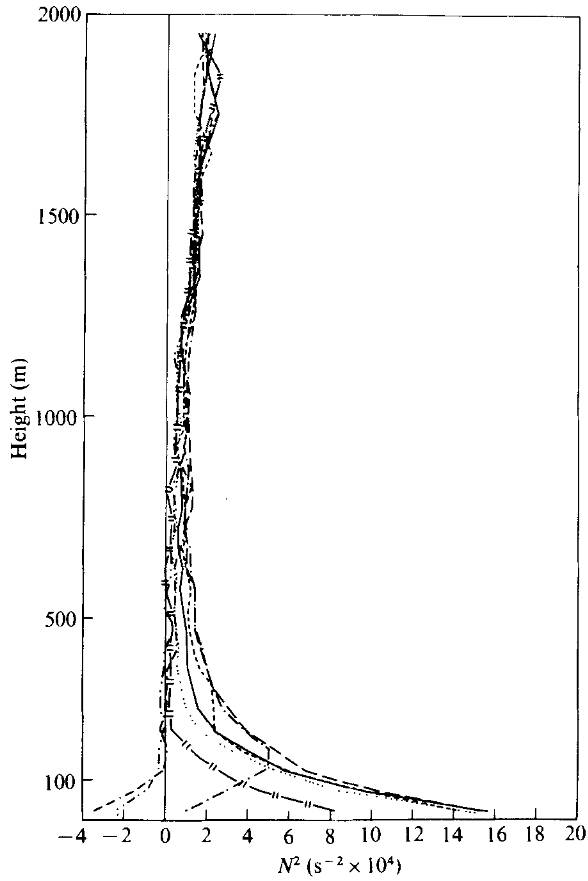

FIGURE 1. The approximately forty-day average of $N^2(z, t)$. ——, 0000; – – –, 0300; — —, 0600; —·—·—, 0900; – –·– –, 1200; —··—, 1500; —॥—, 1800; ·····, 2100.

3. The nocturnal planetary boundary layer

The nocturnal planetary boundary layer (NPBL) has $N^2(z) \approx 0$ near sunset and $N^2(z)$ increasing thereafter. An idea of the typical height–time dependence of N^2 is gained by considering some climatological averages.

The Wangara Experiment (Clarke *et al.* 1971) was used for this purpose because it possessed the advantages of having a high density of observations, flat terrain and moderately dry air. Although many atmospheric variables were measured, here attention is restricted to the temperature and pressure sounding made every three hours during the forty-day period. The measurement of temperature and pressure allows the calculation of potential temperature, which, in turn, allows the computation of N^2.

The number of computed profiles (approximately 8×40) is too large for all to be presented, however investigation of these profiles indicated that N^2 developed during the evening with marked regularity. In view of this, the approximately forty-day average at each observation time (0000, 0300, 0600, 0900, 1200, 1500, 1800, 2100) is meaningful. The results of this calculation are shown in figure 1. The graph indicates that at 1200 the portion of the atmosphere closest to the ground is convectively unstable ($N^2 < 0$), at higher altitudes the atmosphere is neutrally stable ($N \neq 0$) and at even higher altitudes stability is achieved. Similar behaviour is noted for the mid-sounding (1500). The profile shows a very stable region close to the ground at the 1800 sounding, the stability decreasing with height and then increasing to the stability of the free atmosphere. Three hours later (2100), N^2 has doubled its 1800 value up to approximately 500 m, and little temporal change is observed above this height. There is very little time variation of N^2 between 2100 and 0600 and the 0900 sounding shows the previously stable region adjacent to the ground to be severely eroded.

The most widely accepted explanation of the growth of the nocturnal inversion is that due to Blackadar (1957). He states,

> 'Above a height of about a meter the rate of nocturnal cooling is too large to be accounted for by the radiational or conduction fluxes, and it is therefore evident that turbulent transfer is the chief control on the rate of upward propagation of the inversion surface. The cause of the turbulence lies in the large wind shear which develops within the inversion and which is capable of supplying sufficient turbulent energy to overcome the stability.'

That is, the wind-produced turbulence is sustained in such a manner as to produce enhanced transport yet is not of sufficient scale or intensity to destroy the inversion.

Thus it appears that the proposed model is relevant to this situation.

4. An analytical model

The governing equations (2.5)–(2.9) are simplified by observing that the mean wind is mainly horizontal and a function of z only. The effect of wind shear on internal gravity waves, although important, is not germane to this analysis. Hence attention is confined to waves propagating perpendicular to the mean wind vector, since the waves are then ignorant of the mean wind.

The governing equations are obtained by setting $\mathbf{U} = 0$ in (2.5)–(2.9). The familiar procedure of combining these equations into a single equation for the vertical velocity is not affected by the time dependence of N^2 (see, for example, Orlanski 1973). The result of this is

$$\frac{\partial^2}{\partial t^2}\nabla^2 w + N^2(z,t)\frac{\partial^2 w}{\partial x^2} = 0, \tag{4.1}$$

where $\partial/\partial y = 0$ (without loss of generality). A functional form for $N^2(z,t)$ which both allows analytical solutions and simulates well the observed behaviour is

$$N^2(z,t) = \mathcal{N}^2(t)\, N_0^2\, e^{-2z/h}, \tag{4.2}$$

where $N^2(t)$ is dimensionless, N_0^2 is constant and h is the scale depth of N.

Investigated here are perturbations introduced between the times 1500 and 2100. After 2100, N^2 changes little and no variation of wave energy occurs (save for the effects of dissipation). The choice $N^2 = at$ gives fair agreement with the observations, and allows analytical solutions. A more realistic choice is $1 - e^{-t/\tau}$, which gives linear increases at first which then taper off for $t \geqslant \tau$. The analysis ends at sunrise with the beginning of severe thermal convection. Figure 2 displays a comparison between the linear $N^2(t)$ and the climatological profiles of figure 1.

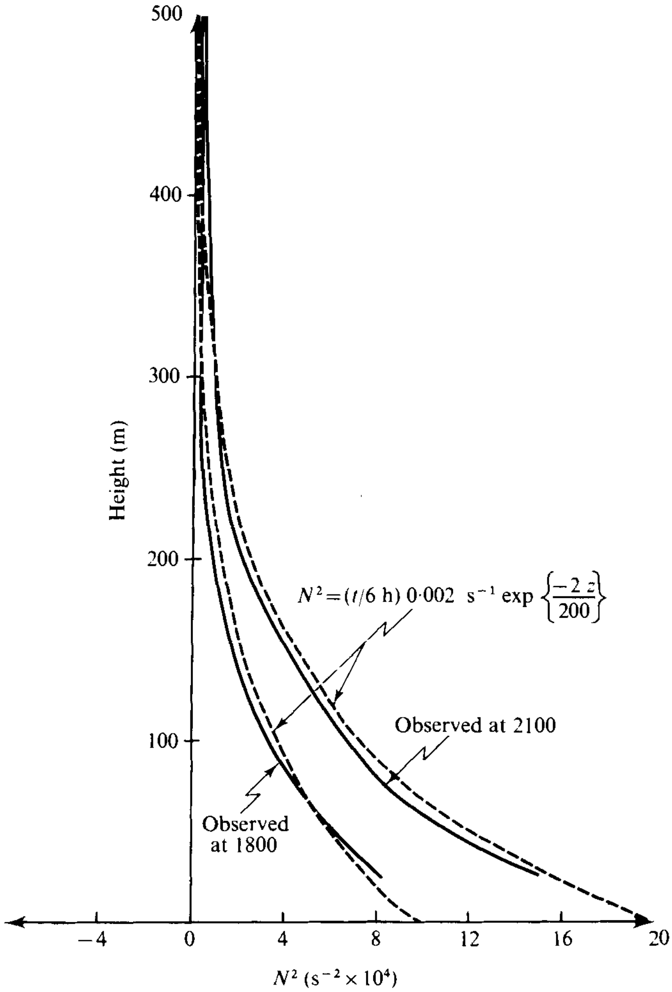

FIGURE 2. Comparison of analytic expression and average N^2 from figure 1.

The boundary conditions are that the vertical velocity should vanish at the ground and at infinity, i.e.

$$w(x,0,t) = 0, \quad w(x,\infty,t) = 0. \tag{4.3}$$

The vertical velocity and acceleration are initially specified as arbitrary† functions of x and z, i.e.

$$w(x,z,0) = A(x,z), \quad \partial w(x,z,0)/\partial t = B(x,z). \tag{4.4}$$

The explicit x dependence may be removed via Fourier transformation, and the result is that the problem specified by (4.1), (4.3) and (4.4) becomes

$$\frac{\partial^4 w}{\partial z^2\,\partial t^2} - k^2\frac{\partial^2 w}{\partial t^2} - k^2\mathscr{N}^2(t)\,N_0^2\,e^{2z/h}w = 0 \tag{4.5}$$

subject to

$$w(0,t) = 0, \quad w(\infty,t) = 0 \tag{4.6}$$

and

$$w(z,0) = A(z), \quad \partial w(z,0)/\partial t = B(z). \tag{4.7}$$

† As long as these are small enough for (4.1) to be valid.

The particular specification (4.2) of $N^2(z,t)$ allows the separation-of-variables procedure in (4.5). Let $w(z,t) = R(z)\,G(t)$. Then (4.5) becomes

$$\frac{\partial^2 R}{\partial z^2}\frac{\partial^2 G}{\partial t^2} - k^2 R\frac{\partial^2 G}{\partial t^2} - k^2\mathcal{N}^2(t)\,N_0^2\,e^{-2z/h}RG = 0. \tag{4.8}$$

Division of (4.8) by $e^{-2z/h}R\,\partial^2 G/\partial t^2$ yields

$$\frac{\partial^2 R/\partial z^2 - k^2 R}{Re^{-2z/h}} = \frac{k^2\mathcal{N}^2(t)\,N_0^2\,G}{\partial^2 G/\partial t^2}. \tag{4.9}$$

Since the left-hand side is a function of t only and the right-hand side is a function of z only, each side must be a constant, $-\gamma^2$, say. Thus the problem of solving the partial differential equation (4.5) is replaced by the problem of solving two ordinary differential equations.

Consider first the boundary-value problem

$$R_{zz} + (\gamma^2 e^{-2z/h} - k^2)\,R = 0 \tag{4.10}$$

subject to

$$R(0) = 0, \quad R(\infty) = 0. \tag{4.11}$$

The transformations $\xi = e^{-z/h}$ and $R(z) = \mathscr{R}(\xi)$ and the definitions $\tilde{\gamma} = \gamma h$ and $\tilde{k} = kh$ convert (4.10) and (4.11) into

$$\xi^2\mathscr{R}_{\xi\xi} + \xi\mathscr{R}_{\xi} + (\tilde{\gamma}^2\xi^2 - \tilde{k}^2)\mathscr{R} = 0 \tag{4.12}$$

and

$$\mathscr{R}(1) = 0, \quad \mathscr{R}(0) = 0, \tag{4.13a,b}$$

respectively.

The general solution to (4.12) is

$$\mathscr{R}(\xi) = c_1 J_{\tilde{k}}(\tilde{\gamma}\xi) + C_2 Y_{\tilde{k}}(\tilde{\gamma}\xi), \tag{4.14}$$

where $J_\nu(\psi)$ and $Y_\nu(\psi)$ are Bessel functions of order ν and argument ψ and c_1 and c_2 are constants. Since $Y_\nu(0)$ is infinite, satisfaction of (4.13*b*) requires $c_2 = 0$. Equation (4.13*a*) is satisfied when

$$J_{\tilde{k}}(\tilde{\gamma}_n) = 0, \quad n = 1, 2, 3, \ldots, \tag{4.15}$$

where the $\tilde{\gamma}_n$ are the zeros of the Bessel function. The most general solution to (4.12) is thus

$$\mathscr{R}(\xi) = \sum_{m=1}^{\infty} \mathscr{R}_m J_{\tilde{k}}(\tilde{\gamma}_m \xi). \tag{4.16}$$

This solution represents waves guided between the ground and the vicinity of the turning point $(h \ln \tilde{\gamma}_n/\tilde{k})$, i.e. the waves propagate horizontally in a modal structure. To obtain the time dependence of the modal amplitudes the equation resulting from the right-hand side of (4.9) is considered:

$$\frac{\partial^2 G_n}{\partial t^2} + \left(\frac{\tilde{k}\mathcal{N}N_0}{\tilde{\gamma}}\right)^2 G_n = 0. \tag{4.17}$$

The orthogonality and completeness of the set of Bessel functions allow the initial conditions to be written as

$$G_n(0) = \mathscr{A}_n/\mathscr{R}_n, \quad \partial G_n(0)/\partial t = \mathscr{B}_n/\mathscr{R}_n, \tag{4.18}$$

where $$\{\mathscr{A}_n, \mathscr{B}_n\} = \frac{2}{\{J_k'(\tilde{\gamma}_n)\}^2} \int_0^1 \{\mathscr{A}(\xi), \mathscr{B}(\xi)\}\, \xi J_k(\tilde{\gamma}\xi)\, d\xi. \tag{4.19}$$

Hence the general solution to (4.17) is

$$G_n(t) = \left\{\left(G_{1n}(t)\frac{\partial G_{2n}}{\partial t}(0) - G_{2n}(t)\frac{\partial G_{1n}}{\partial t}(0)\right)\mathscr{A}_n + \left(G_{1n}(0)\frac{\partial G_{2n}}{\partial t}(t) - G_{2n}(0)\frac{\partial G_{1n}}{\partial t}(0)\right)\mathscr{B}_n\right\} \Big/ W[G_{1n}(0), G_{2n}(0)], \tag{4.20}$$

where $W(x,y) =$ Wronskian of $(x,y) = (xy_t - yx_t)_{t=0}$.

A specific example is the case $\mathscr{N}^2(t) = a^2t$. The general solution to (4.17) is

$$G_{1n,\,2n}(t) = \left(\frac{a\tilde{k}N_0}{\tilde{\gamma}_n}\right)^{\frac{1}{3}} t^{\frac{1}{2}} J_{\mp\frac{1}{3}}\left(\frac{2}{3}\frac{2\tilde{k}N_0}{\tilde{\gamma}_n}t^{\frac{3}{2}}\right). \tag{4.21}$$

The late-time behaviour of these solutions provides a consistency check on the results of §2. For large t,

$$G_{1n,\,2n} \sim t^{-\frac{1}{4}} \cos\left(\frac{2}{3}\frac{a\tilde{k}N_0}{\tilde{\gamma}_n}t^{\frac{3}{2}} - \frac{\pi}{4} \pm \frac{\pi}{6}\right), \tag{4.22}$$

a result which could have been obtained by direct application of the WKB approximation to (4.17). The horizontal velocity exhibits the same time dependence since the time-varying N produces no changes in wavelength. Therefore the local period average of the kinetic energy behaves as $t^{-\frac{1}{2}}$. The local period average of the potential energy behaves likewise. Hence the total energy exhibits a $t^{-\frac{1}{2}}$ dependence. The local frequency is obtained by taking the time derivative of the cosine's argument:

$$\omega_n = (a\tilde{k}N_0/\tilde{\gamma}_n)\, t^{\frac{1}{2}}. \tag{4.23}$$

It is obvious that the product of the local energy density and frequency is constant.

The wave energy should change only as long as N^2 changes. A function

$$\mathscr{N}^2(t) = 1 - e^{-t/\tau}$$

is chosen to simulate the observed behaviour that after initial (first few hours) increases in $N^2(z,t)$ subsequent changes take place much more slowly. The general solution to (4.17) can be expressed in terms of modified Bessel functions of imaginary order. However, it is more convenient to consider the results of a numerical integration. Figure 3 displays two curves representing oscillations in the first and third modes (in both cases $G(0) = 1$ and $G_t(0) = 0$). The graph indicates that the higher-frequency oscillation is more susceptible to the effects of increasing stratification. Notice that just before $t = \tau\,(4\,\text{h})$ the wave amplitudes and frequencies cease to be seriously modulated.

5. Summary

There exist a number of geophysical situations where the Brunt–Väisälä frequency exhibits strong temporal variation. The cumulative effect of a periodically varying N^2 is instability of the wave motion (Orlanski 1973). However, local variations of N also affect the wave energy.

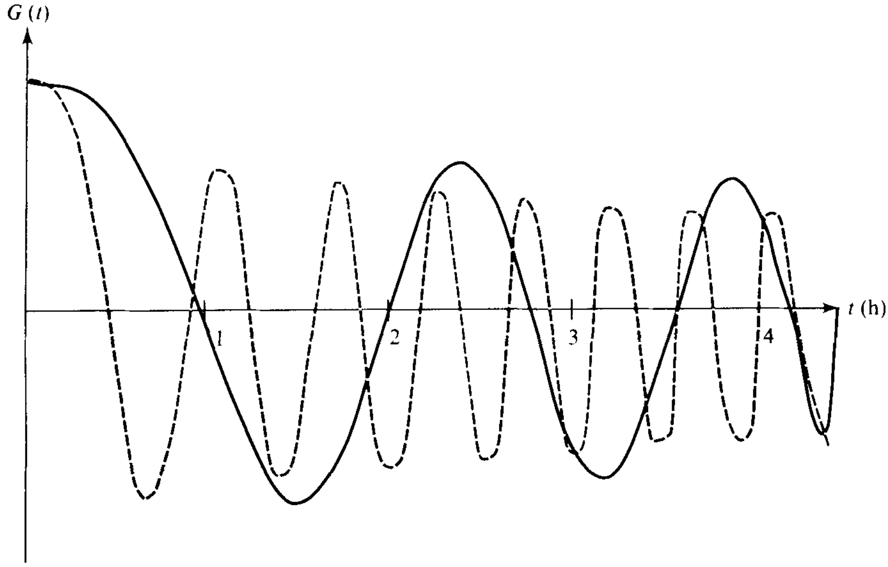

FIGURE 3. $G(t)$ *vs.* t. —, $n = 3$; – – –, $n = 1$.

The asymptotic theory for slowly varying $N^2(t)$ (Garrett 1968) was reviewed and certain aspects available from that analysis were emphasized. When $N^2(t)$ varies owing to a certain type of irreversible process, the conservation law obtained is very different; it is found that the product of energy density and frequency is conserved.

The NPBL is an example where this process could be important. The full initial/boundary-value problem using an $N^2(z, t)$ similar to some climatological means is presented and solved.

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